INFINITE-PERIOD ECONOMY

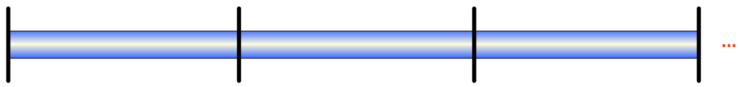
FEBRUARY 25, 2009

BASICS

- **Modern workhorse macroeconomic models feature an infinite number of periods**
  - **A more realistic (?) view of time**
- **Especially useful for thinking about asset accumulation and asset pricing**
  - **The intersection of modern macro theory and modern finance theory**
- **Here, assume just one real asset – opening up the "black box"**
  - **Call it a "stock" – i.e., a share in the S&P 500**
  - **(In Chapter 14, two nominal assets: bonds and money)**
- **Index time periods by arbitrary indexes *t*, *t*+1, *t*+2, etc.**
  - **Important: all of our analysis will be conducted from the perspective of the very beginning of period *t*...**
  - **...so an "infinite future" (period *t*+1, period, *t*+2, period *t*+3, ...) for which to save**

BASICS

- **Timeline of events**



- **Notation**
  - c_t: consumption in period *t*
  - P_t: nominal price of consumption in period *t*
  - Y_t: nominal income in period *t* ("falls from the sky")
  - a_{t-1}: real wealth (stock) holdings at beginning of period *t* / end of period *t*-1
  - S_t: nominal price of a unit of stock in period *t*
  - D_t: nominal dividend paid in period *t* by each unit of stock held at the start of *t*

  The "definining features" of stock

  - π_{t+1}: net inflation rate between period *t* and period *t*+1

$$\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t}\left(= \frac{P_{t+1}}{P_t} - 1\right)$$

  - y_t: real income in period *t* (= Y_t / P_t)

a_{t-1}

Economic events during period *t*: income, consumption, savings

a_t

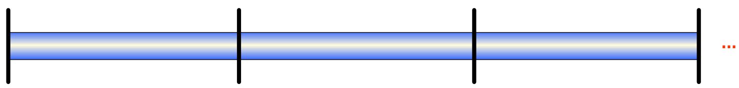
Economic period *t* consump

Period *t*

Peri

BASICS

- **Timeline of events**



- **Notation**
  - c_{t+1}: consumption in period *t*+1
  - P_{t+1}: nominal price of consumption in period *t*+1
  - Y_{t+1}: nominal income in period *t*+1 ("falls from the sky")
  - a_t: real wealth (stock) holdings at beginning of period *t*+1 / end of period *t*
  - S_{t+1}: nominal price of a unit of stock in period *t*+1
  - D_{t+1}: nominal dividend paid in period *t* by each unit of stock held at the start of *t*+1

  The "definining features" of stock

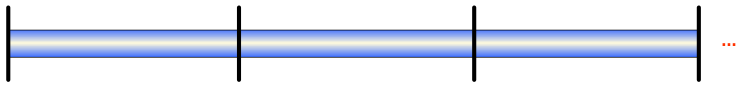
  - π_{t+2}: net inflation rate between period *t*+1 and period *t*+2

$$\pi_{t+2} = \frac{P_{t+2} - P_{t+1}}{P_{t+1}}\left(= \frac{P_{t+2}}{P_{t+1}} - 1\right)$$

  - y_{t+1}: real income in period *t*+1 (= Y_{t+1} / P_{t+1})

BASICS

- **Timeline of events**



- **Notation**
  - **And so on for period *t*+2, *t*+3, etc…**

a_{t-1}

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Economic period *t* consump

Period *t*

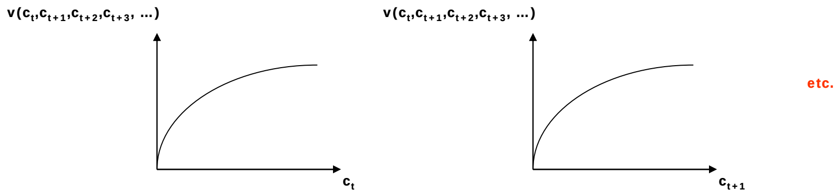
Peri

SUBJECTIVE DISCOUNT FACTOR

- **Infinite number of periods a more serious view of time**
- **Impatience potentially an issue when taking a serious view of time**
- **Individuals (i.e., consumers) are impatient**
  - **All else equal, would rather have outcome *X* today than identical outcome *X* at some future date**
  - **An introspective statement about the world**
  - **An empirical statement about the world**
- **Subjective discount factor**
  - **A simple model of consumer impatience**
  - **β (a number between zero and one) measures impatience**
    - **The lower is β, the less does individual value future utility**
  - **Simple assumption about how "impatience" builds up over time**
    - **Multiplicatively: i.e., discount one period ahead by β, discount two periods ahead by β^2, discount three periods ahead by β^3, etc.**
    - **Do individuals' impatience really build up over time in this way?...limited empirical evidence so really don't know…**

UTILITY

- Preferences $v(c_t, c_{t+1}, c_{t+2}, \ldots)$ with all the "usual properties"
  - Lifetime utility function
  - Strictly increasing in $c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots$
  - Diminishing marginal utility in $c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots$



- Lifetime utility function additively-separable across time (a simplifying assumption), starting at time *t*

$$v(c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots) = u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \ldots$$

- Utility side of infinite-period model no different than Chapter 1 model – except no longer possible to represent graphically…

BUDGET CONSTRAINT(S)

- Suppose again *Y* "falls from the sky"
  - Y_t in period t, Y_{t+1} in period $t+1$, Y_{t+2} in period $t+2$, etc.

- Need infinite budget constraints to describe economic opportunities and possibilities
  - One for each period

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  - **Period-*t* budget constraint**

$$\underbrace{P_t c_t + S_t a_t}_{} = \underbrace{Y_t + S_t a_{t-1} + D_t a_{t-1}}_{}$$

Total expenditure in period *t*: period-*t* consumption + wealth to *carry into period t*+1

Total income in period *t*: period-*t Y* + income from stock-holdings *carried into period t* (has value S_t and pays dividend D_t)

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Total expenditure in period *t*+1: period-*t*+1 consumption + wealth to *carry into period t*+2

Total income in period *t*+1: period-*t*+1 *Y* + income from stock-holdings *carried into period t*+1 (has value S_{t+1} and pays dividend D_{t+1})

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can rewrite as

$$P_t c_t + S_t (a_t - a_{t-1}) = Y_t + D_t a_{t-1}$$

Savings during period *t* (a flow)

Dividend income during period *t* (a flow)

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$$P_{t+1} c_{t+1} + S_{t+1} a_{t+1} = Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t$$

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And identical-looking budget constraints for *t*+2, *t*+3, *t*+4, etc…

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Sequential formulation highlights the role of stock holdings (a_t) between period t and period $t+1$**
  - **Accords better with the explicit timing of economic events than the lifetime approach…**
  - **…but yields the same result**
  - **Advantage: allows us to think about interaction between asset prices and macroeconomic events (intersection of finance theory and macro theory)**

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- **Apply Lagrange tools to consumption-savings optimization**
- **Objective function: $v(c_t, c_{t+1}, c_{t+2}, \ldots)$**
- **Constraints:**
  - **Period-t budget constraint:** $Y_t + S_t a_{t-1} + D_t a_{t-1} - P_t c_t - S_t a_t = 0$
  - **Period-$t+1$ budget constraint:** $Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t - P_{t+1} c_{t+1} - S_{t+1} a_{t+1} = 0$
  - **Period-$t+2$ budget constraint:** $Y_{t+2} + S_{t+2} a_{t+1} + D_{t+2} a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2} = 0$
  - **etc…**

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- Apply Lagrange tools to consumption-savings optimization
- Objective function: $v(c_t, c_{t+1}, c_{t+2}, \ldots)$
- Constraints:
  - INFINITE constraints:
  - Period-t budget constraint: $Y_t + S_t a_{t-1} + D_t a_{t-1} - P_t c_t - S_t a_t = 0$
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  - etc…
- Sequential Lagrange formulation requires infinite multipliers

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Step 1:** Construct Lagrange function (starting from t)

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$$u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})+\beta^3 u(c_{t+3})+\ldots$$

First the lifetime utility function….

Infinite-Period Model: Sequential Formulation

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$$u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})+\beta^3 u(c_{t+3})+\ldots$$
$$+\lambda_t\left[Y_t+(S_t+D_t)a_{t-1}-P_t c_t-S_t a_t\right]$$

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$$+\beta\lambda_{t+1}\left[Y_{t+1}+(S_{t+1}+D_{t+1})a_t-P_{t+1}c_{t+1}-S_{t+1}a_{t+1}\right]$$ …then the period *t*+1 constraint…

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$$+\beta^2\lambda_{t+2}\left[Y_{t+2}+(S_{t+2}+D_{t+2})a_{t+1}-P_{t+2}c_{t+2}-S_{t+2}a_{t+2}\right]$$ …then the period *t*+2 constraint…

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$$+\beta^3\lambda_{t+3}\left[Y_{t+3}+(S_{t+3}+D_{t+3})a_{t+2}-P_{t+3}c_{t+3}-S_{t+3}a_{t+3}\right]$$ ...then the period *t*+3 constraint...

Infinite-Period Model: Sequential Formulation

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$$+\ldots$$ Infinite number of terms

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Step 1: Construct Lagrange function (starting from t)**

IMPORTANT: Discount factor β multiplies ***both*** future utility ***and*** future budget constraints

Everything (utility ***and*** income) about the future is discounted

$$u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})+\beta^3 u(c_{t+3})+\dots$$ First the lifetime utility function….

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- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

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First the lifetime utility function….
…then the period *t* constraint…
…then the period *t*+1 constraint…
…then the period *t*+2 constraint…
…then the period *t*+3 constraint…
Infinite number of terms

- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

with respect to c_t:

with respect to a_t:

with respect to c_{t+1}:

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

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Infinite number of terms

- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

Identical except for time subscripts

with respect to c_t: $u'(c_t) - \lambda_t P_t = 0$ Equation 1

with respect to a_t: $-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$ Equation 2

with respect to c_{t+1}: $\cancel{\beta} u'(c_{t+1}) - \cancel{\beta}\lambda_{t+1}P_{t+1} = 0$ Equation 3

THE BASICS OF ASSET PRICING

$$u'(c_t) - \lambda_t P_t = 0$$ Equation 1

$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ Equation 2

$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0$$ Equation 3

- **Equation 2 →** $S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$ BASIC ASSET-PRICING EQUATION

Finance Fundamentals

THE BASICS OF ASSET PRICING

$$u'(c_t) - \lambda_t P_t = 0$$ Equation 1

$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ Equation 2

$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0$$ Equation 3

- **Equation 2 →** $S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$ BASIC ASSET-PRICING EQUATION

Period-*t* stock price = Pricing kernel x Future return

Two components:
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  - **Empirical models of kernels**

- **Pricing kernel where macro theory and finance theory intersect**

Finance Fundamentals

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- **Pricing kernel where macro theory and finance theory intersect**
  - **Solve equations 1 and 3 for λ_t and λ_{t+1}**
  - **Insert in asset-pricing equation**

Macroeconomic Events Affect Asset Prices

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

- Consumption across time (c_t and c_{t+1}) affects stock prices
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- ***Wall Street Journal* page A1, Feb. 21, 2008:**

  "Stocks [prices] fell on the [higher-than-expected] inflation reading."

  Interpretation: Increase in π_t (which typically would also be associated with increase in π_{t+1}) led to fall in S_t.

Macro-Finance Connections

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- ***Wall Street Journal* page A1, Oct. 2, 2008:**

  "Stock prices fell on fears of short-run weakness in the economy."

  Interpretation: A decrease in c_t (induced by the "credit crunch") makes stocks *right now* (i.e., in period *t*) a less attractive asset – demand for it falls, hence its price (S_t) falls.

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- **Direction of causality?...**

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VIEW AS A CONSUMPTION-SAVINGS OPTIMALITY CONDITION →

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$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

↓ **Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side**

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i.e., ratio of marginal utilities → **MRS between period t consumption and period $t+1$ consumption**

Some sort of price ratio…

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Consumption-Savings View

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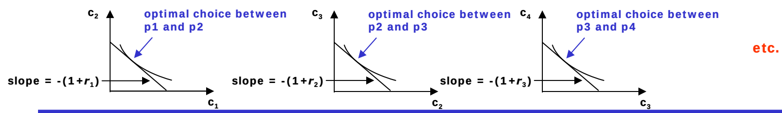
- Recover Chapter 3 & 4 model by setting t =1 and β = 1
- Infinite-period model is sequence of overlapping two-period models



A LONG-RUN THEORY OF MACRO

- Consumption-savings optimality condition at the heart of modern macro models
  - Emphasize the dynamic nature of aggregate economic events
  - Foundation for understanding the periodic ups and downs ("business cycles") of the economy
  - (Chapter 13: business cycle models)

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↓ NEXT: Impose "steady state" and examine long-run relationship between interest rates and consumer impatience

$$\frac{1}{\beta} = 1 + r$$