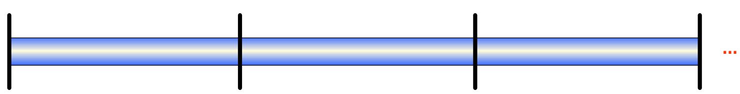
INFINITE-PERIOD ECONOMY (CONTINUED)

MARCH 2, 2009

BASICS

- Timeline of events



- Notation
  - c_t: consumption in period t
  - P_t: nominal price of consumption in period t
  - Y_t: nominal income in period t ("falls from the sky")
  - a_{t-1}: real wealth (stock) holdings at beginning of period t / end of period t-1
  - S_t: nominal price of a unit of stock in period t
  - D_t: nominal dividend paid in period t by each unit of stock held at the start of t

  The "definining features" of stock

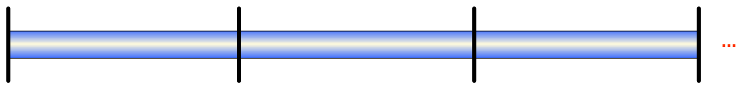
  - π_{t+1}: net inflation rate between period t and period t+1

$$\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t} \left(= \frac{P_{t+1}}{P_t} - 1 \right)$$

  - y_t: real income in period t (= Y_t / P_t)

BASICS

- **Timeline of events**



- **Notation**
  - c_{t+1}: consumption in period $t+1$
  - P_{t+1}: nominal price of consumption in period $t+1$
  - Y_{t+1}: nominal income in period $t+1$ ("falls from the sky")
  - a_t: real wealth (stock) holdings at beginning of period $t+1$ / end of period t
  - S_{t+1}: nominal price of a unit of stock in period $t+1$
  - D_{t+1}: nominal dividend paid in period t by each unit of stock held at the start of $t+1$
  - π_{t+2}: net inflation rate between period $t+1$ and period $t+2$

The "definining features" of stock

$$\pi_{t+2} = \frac{P_{t+2} - P_{t+1}}{P_{t+1}} \left(= \frac{P_{t+2}}{P_{t+1}} - 1 \right)$$

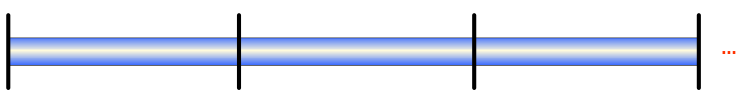
  - y_{t+1}: real income in period $t+1$ ($= Y_{t+1} / P_{t+1}$)

a_{t-1} Economic events during period t: income, consumption, savings a_t Economic period t consump

Period t Peri

BASICS

- **Timeline of events**



- **Notation**
  - And so on for period $t+2$, $t+3$, etc…

BUDGET CONSTRAINT(S)

- **Suppose again *Y* "falls from the sky"**
  - Y_t in period t, Y_{t+1} in period $t+1$, Y_{t+2} in period $t+2$, etc.
- **Need infinite budget constraints to describe economic opportunities and possibilities**
  - One for each period
  - Period-t budget constraint

$$P_t c_t + S_t a_t = Y_t + S_t a_{t-1} + D_t a_{t-1} \quad \text{can rewrite as} \quad P_t c_t + S_t (a_t - a_{t-1}) = Y_t + D_t a_{t-1}$$

Total expenditure in period t: period-t consumption + wealth to *carry into period* $t+1$

Total income in period t: period-t Y + income from stock-holdings *carried into period* t (has value S_t and pays dividend D_t)

Savings during period t (a flow)

Dividend income during period t (a flow)

  - Period $t+1$ budget constraint

$$P_{t+1} c_{t+1} + S_{t+1} a_{t+1} = Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t \quad \text{can rewrite as} \quad P_{t+1} c_{t+1} + S_{t+1}(a_{t+1} - a_t) = Y_{t+1} + D_{t+1} a_t$$

Total expenditure in period $t+1$: period-$t+1$ consumption + wealth to *carry into period* $t+2$

Total income in period $t+1$: period-$t+1$ Y + income from stock-holdings *carried into period* $t+1$ (has value S_{t+1} and pays dividend D_{t+1})

Savings during period $t+1$ (a flow)

Dividend income during period $t+1$ (a flow)

And identical-looking budget constraints for $t+2$, $t+3$, $t+4$, etc…

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Sequential formulation highlights the role of stock holdings (a_t) between period t and period $t+1$**
  - Accords better with the explicit timing of economic events than the lifetime approach…
  - …but yields the same result
  - Advantage: allows us to think about interaction between asset prices and macroeconomic events (intersection of finance theory and macro theory)
- **Apply Lagrange tools to consumption-savings optimization**
- **Objective function: $v(c_t, c_{t+1}, c_{t+2}, \ldots)$**
- **Constraints:**
  - Period-t budget constraint: $Y_t + S_t a_{t-1} + D_t a_{t-1} - P_t c_t - S_t a_t = 0$
  - Period-$t+1$ budget constraint: $Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t - P_{t+1} c_{t+1} - S_{t+1} a_{t+1} = 0$
  - Period-$t+2$ budget constraint: $Y_{t+2} + S_{t+2} a_{t+1} + D_{t+2} a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2} = 0$
  - etc…

INFINITE constraints

- **Sequential Lagrange formulation requires infinite multipliers**

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Step 1: Construct Lagrange function (starting from t)**

IMPORTANT: Discount factor β multiplies ***both*** future utility ***and*** future budget constraints

Everything (utility ***and*** income) about the future is discounted

$$u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})+\beta^3 u(c_{t+3})+\ldots$$ First the lifetime utility function....

$$+\lambda_t\left[Y_t+(S_t+D_t)a_{t-1}-P_t c_t-S_t a_t\right]$$ ...then the period t constraint...

$$+\beta\lambda_{t+1}\left[Y_{t+1}+(S_{t+1}+D_{t+1})a_t-P_{t+1}c_{t+1}-S_{t+1}a_{t+1}\right]$$ ...then the period $t+1$ constraint...

$$+\beta^2\lambda_{t+2}\left[Y_{t+2}+(S_{t+2}+D_{t+2})a_{t+1}-P_{t+2}c_{t+2}-S_{t+2}a_{t+2}\right]$$ ...then the period $t+2$ constraint...

$$+\beta^3\lambda_{t+3}\left[Y_{t+3}+(S_{t+3}+D_{t+3})a_{t+2}-P_{t+3}c_{t+3}-S_{t+3}a_{t+3}\right]$$ ...then the period $t+3$ constraint...

$$+\ldots$$ Infinite number of terms

- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

with respect to c_t:

with respect to a_t:

with respect to c_{t+1}:

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$$+\ldots$$ Infinite number of terms

- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

Identical except for time subscripts

with respect to c_t: $u'(c_t)-\lambda_t P_t=0$ Equation 1

with respect to a_t: $-\lambda_t S_t+\beta\lambda_{t+1}(S_{t+1}+D_{t+1})=0$ Equation 2 – the basis for asset pricing theories

with respect to c_{t+1}: $\beta u'(c_{t+1})-\beta\lambda_{t+1}P_{t+1}=0$ Equation 3

The Basics of Asset Pricing

$$u'(c_t) - \lambda_t P_t = 0 \quad \text{Equation 1}$$
$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0 \quad \text{Equation 2}$$
$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0 \quad \text{Equation 3}$$

- **Equation 2 →** $S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$ — BASIC ASSET-PRICING EQUATION

Period-*t* stock price = Pricing kernel x Future return

Two components:
1. Future price of stock
2. Future dividend payment

- **Much of finance theory concerned with pricing kernel**
  - **Theoretical properties**
  - **Empirical models of kernels**

- **Pricing kernel where macro theory and finance theory intersect**
  - **Solve equations 1 and 3 for λ_t and λ_{t+1}**
  - **Insert in asset-pricing equation**

Macroeconomic Events Affect Asset Prices

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

- **Consumption across time (c_t and c_{t+1}) affects stock prices**
  - **Fluctuations over time in aggregate consumption impact S_t**

MACROECONOMIC EVENTS AFFECT ASSET PRICES

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↓ **Using definition of inflation: $1+\pi_{t+1} = P_{t+1} / P_t$**

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

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  - **Fluctuations over time in inflation impact S_t**
- ***Wall Street Journal* page A1, Feb. 21, 2008:**

  "Stocks [prices] fell on the [higher-than-expected] inflation reading."

  Interpretation: Increase in π_t (which typically would also be associated with increase in π_{t+1}) led to fall in S_t.

MACROECONOMIC EVENTS AFFECT ASSET PRICES

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  - **Fluctuations over time in inflation impact S_t**
- ***Wall Street Journal* page A1, Oct. 3, 2008:**

  "Stock prices fell on fears of short-run weakness in the economy."

  Interpretation: A decrease in c_t (induced by the "credit crunch") makes stocks *right now* (i.e., in period t) a less attractive asset – demand for it falls, hence its price (S_t) falls.

MACROECONOMIC EVENTS AFFECT ASSET PRICES

Combining equations 1, 2, and 3 →

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1}+D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

↓ Using definition of inflation: $1+\pi_{t+1} = P_{t+1} / P_t$

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1}+D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

VIEW AS A CONSUMPTION-SAVINGS OPTIMALITY CONDITION →

- **Consumption across time (c_t and c_{t+1}) affects stock prices**
  - **Fluctuations over time in aggregate consumption impact S_t**
- **Inflation affects stock prices**
  - **Fluctuations over time in inflation impact S_t**
- **ANY factor (monetary policy, fiscal policy, globalization, etc.) that affects inflation and GDP in principle impacts stock/asset markets**
- **Direction of causality?...**

Consumer Optimization

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

↓ **Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side**

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \left(\frac{S_{t+1} + D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)$$

Consumption-Savings View

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i.e., ratio of marginal utilities → **MRS between period t consumption and period $t+1$ consumption**

Some sort of price ratio…

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CONSUMPTION-SAVINGS OPTIMALITY CONDITION

i.e., ratio of marginal utilities → MRS between period t consumption and period $t+1$ consumption

Analogy with Chapters 3 & 4: must be $(1+r_t)$ ← Recall real interest rate is a ***price***

Consumption-Savings View

Consumer Optimization

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- **Recover Chapter 3 & 4 model by setting t = 1 and β = 1**

CONSUMER OPTIMIZATION

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Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side

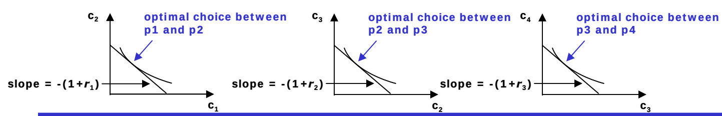
$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \left(\frac{S_{t+1}+D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)$$

CONSUMPTION-SAVINGS OPTIMALITY CONDITION

i.e., ratio of marginal utilities → MRS between period t consumption and period $t+1$ consumption

Analogy with Chapters 3 & 4: must be $(1+r_t)$ ← Recall real interest rate is a ***price***

- Recover Chapter 3 & 4 framework by setting t =1 and β = 1
- Infinite-period framework is sequence of ***overlapping*** two-period frameworks



etc.

A LONG-RUN THEORY OF MACRO

- Consumption-savings optimality condition at the heart of modern macro frameworks
  - Emphasizes the dynamic nature of aggregate economic events
  - Foundation for understanding the periodic ups and downs ("business cycles") of the economy
  - (Chapter 13: business cycle theory)

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t$$

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↓ NEXT: Impose "steady state" and examine long-run relationship between interest rates and consumer impatience

$$\frac{1}{\beta} = 1 + r$$