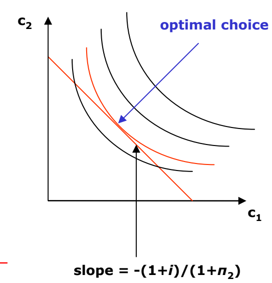
CONSUMPTION-SAVINGS MODEL
(CONTINUED)

FEBRUARY 11, 2009

CONSUMER OPTIMIZATION

- **Consumer's decision problem:** **maximize lifetime utility subject to lifetime budget constraint – bring together both cost side and benefit side**
  - **Choose c_1 and c_2 subject to** $P_1 c_1 + \frac{P_2 c_2}{1+i} = Y_1 + \frac{Y_2}{1+i}$
  - **Plot budget line**
  - **Superimpose indifference map**



- **At the optimal choice**

CONSUMPTION-SAVINGS OPTIMALITY CONDITION

- A key building block of modern macro models

$$\underbrace{\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)}}_{\text{MRS (between consumption in consecutive time periods)}} = \underbrace{\frac{1+i}{1+\pi_2}}_{\text{price ratio (across consecutive time periods)}}$$

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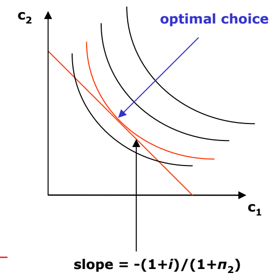

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MRS (between consumption in consecutive time periods) — **price ratio (across consecutive time periods)**

Derive consumption-leisure optimality condition using Lagrange analysis

LAGRANGE ANALYSIS

- **Apply Lagrange tools to consumption-savings optimization**
- **Objective function: $u(c_1,c_2)$**
- **Constraint:** $g(c_1, c_2) = Y_1 + \frac{Y_2}{1+i} - P_1c_1 - \frac{P_2c_2}{1+i} = 0$

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$$L(c_1,c_2,\lambda) = u(c_1,c_2) + \lambda\left[Y_1 + \frac{Y_2}{1+i} - P_1c_1 - \frac{P_2c_2}{1+i}\right]$$

- **Step 2:** **Compute first-order conditions with respect to** c_1, c_2, λ

The Mathematics of the Consumption-Savings Model

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- **Step 3:** **Solve (with focus on eliminating multiplier)**

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CONSUMPTION-SAVINGS OPTIMALITY CONDITION

SAVINGS AND ASSET POSITIONS

- Definition: A consumer's savings during a given time period is the ***change*** **in his wealth** during that time period
- Assets/wealth (whether positive or negative) are a means for "transferring income over time"

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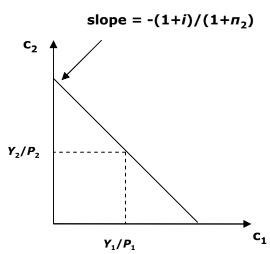

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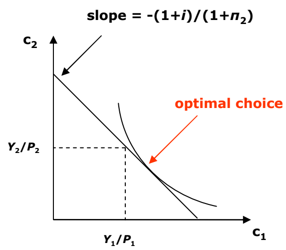

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Optimal $c_1 > Y_1/P_1$ → thus consumer is in debt at the end of period 1 (i.e., period-1 *savings* is negative)

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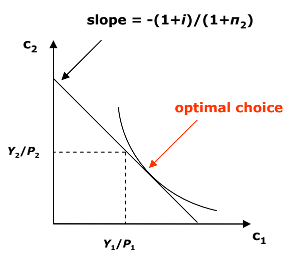

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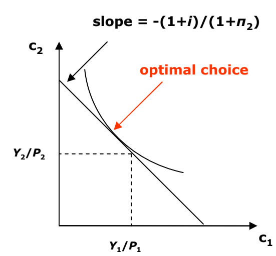

Optimal $c_1 < Y_1/P_1$ → thus consumer has positive wealth at the end of period 1 (i.e., period-1 *savings* is positive)

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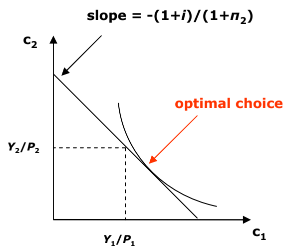

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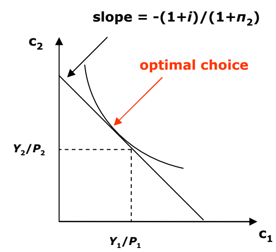

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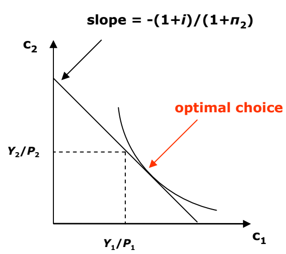
ASSESSING THE CREDIT CRUNCH



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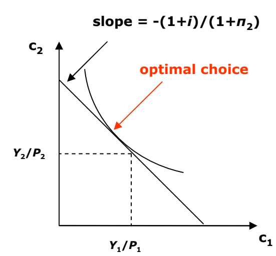

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Use this framework to analyze the channel by which financial market problems have been affecting consumption activity

FISHER EQUATION

- **Nominal interest rate – measured in dollars**
- **Real interest rate – measured in goods**

- **Fisher equation: a link between the nominal interest rate, inflation rate, and real interest rate**
  - **"Strips out the effect of inflation"**

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More useful for our theoretical model

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$$\cancel{1}+r+\pi+r\cancel{\pi}=\cancel{1}+i$$

≈ 0

In developed countries, *r* and *π* are both generally small → *rπ* ≈ 0

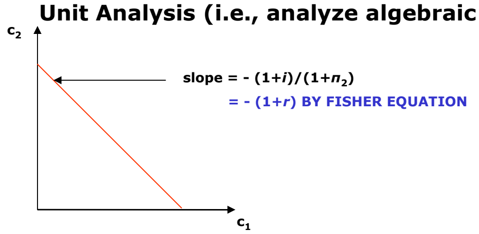
$$r=i-\pi$$

A useful rule of thumb

Real Interest Rate

- *r* a key variable for macroeconomic analysis

- Unit Analysis (i.e., analyze algebraic units of variables)



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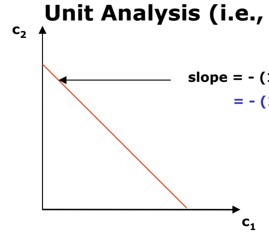

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= $-(1+r)$ BY FISHER EQUATION

Slope measures how much c_2 must be given up in order to obtain one more unit of c_1 ("rise over run") when borrowing or lending at market prices

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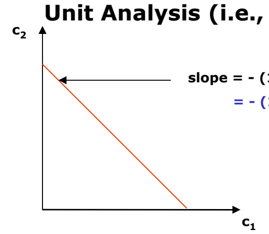

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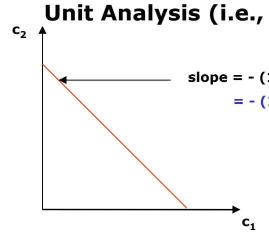

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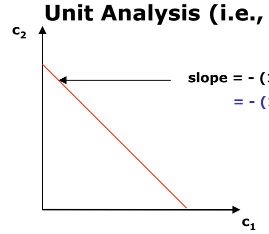

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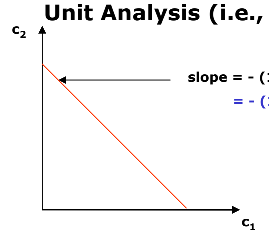

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- **Economic decisions depend on *real* interest rates (*r*), not nominal interest rates (*i*)**
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- Currently: nominal *i* (short-term) ≈ 0%, π ≈ 0% (CPI measure)
  - Real interest rate ≈ 0 right now…

TWO-PERIOD MODEL IN REAL TERMS

- Depending on application, may be useful to work with model (independent of lifetime vs. sequential approach) in nominal terms or in real terms

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LBC in nominal terms (assuming $A_0 = 0$)

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Use definitions: $y_1 = Y_1/P_1$, $y_2 = Y_2/P_2$, and $1+\pi_2 = P_2/P_1$

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Use Fisher equation: $(1+\pi_2)/(1+i) = 1/(1+r)$

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LBC in real terms (assuming $A_0 = 0$)

Maximize $u(c_1,c_2)$ subject to the real LBC → identical consumption-savings optimality condition (details in recitations)

CONSUMPTION-SAVINGS OPTIMALITY CONDITION

- **Emphasizing *i* and *π***

$$\frac{u_1(c_1^*,c_2^*)}{u_2(c_1^*,c_2^*)} = \frac{1+i}{1+\pi}$$

↓ Fisher equation

- **Emphasizing *r***

$$\frac{u_1(c_1^*,c_2^*)}{u_2(c_1^*,c_2^*)} = 1+r$$

- **Can also analyze two-period model sequentially, rather than from a lifetime perspective**

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Sequential formulation highlights the role of net wealth (A_1) between period 1 and period 2**
  - **Accords better with the explicit timing of economic events than the lifetime approach…**
  - **…but yields the same result**
  - **Advantage: allows us to think about interaction between asset prices and macroeconomic events (intersection of finance theory and macro theory in Chapter 8)**

Consumption-Savings Model: Sequential Formulation

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- **Apply Lagrange tools to consumption savings optimization**
- **Objective function: $u(c_1,c_2)$**
- **Constraints:**
  - **Period 1 budget constraint:** $Y_1 + (1+i)A_0 - P_1c_1 - A_1 = 0$
  - **Period 2 budget constraint:** $Y_2 + (1+i)A_1 - P_2c_2 - A_2 = 0$

  } **TWO constraints**

- **Sequential Lagrange formulation requires two multipliers**
  - **See Math Refresher, Chapter -1**