

Economics 602
Macroeconomic Theory and Policy
Midterm Exam Suggested Solutions
Professor Sanjay Chugh
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NAME: _____

The Exam has a total of four (4) questions and pages numbered one (1) through fourteen (14). Each question's total number of points is shown below. You are to answer all questions in the spaces provided.

You may use a non-programmable calculator.

Question 1	/ 20
Question 2	/ 25
Question 3	/ 25
Question 4	/ 30
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TOTAL	/100

Question 1: Consumption and Savings in the Two-Period Economy (20 points). Consider a two-period economy (with no government), in which the representative consumer has no control over his income. The lifetime utility function of the representative consumer is $u(c_1, c_2) = \ln(c_1) + \ln(c_2)$, where \ln stands for the natural logarithm. We will work here in purely real terms: suppose the consumer's **present discounted value of ALL lifetime real income is 26**. Also suppose the consumer begins period 1 with zero assets.

- a. **(13 points)** Set up the lifetime Lagrangian formulation of the consumer's problem, in order to answer the following: i) is it possible to numerically compute the consumer's optimal choice of consumption in period 1? If so, compute it; if not, explain why not. ii) is it possible to numerically compute the consumer's optimal choice of consumption in period 2? If so, compute it; if not, explain why not. iii) is it possible to numerically compute the consumer's real asset position at the end of period 1? If so, compute it; if not, explain why not.

Solution: We know that with zero initial assets, the LBC of the consumer is

$$c_1 + \frac{c_2}{1+r_1} = y_1 + \frac{y_2}{1+r_1},$$

where the notation is standard from class. The Lagrangian is thus

$$u(c_1, c_2) + \lambda \left[y_1 + \frac{y_2}{1+r_1} - c_1 - \frac{c_2}{1+r_1} \right],$$

where λ of course is the Lagrange multiplier (note there's only one multiplier since this is the lifetime formulation of the problem not the sequential formulation of the problem). The first-order conditions with respect to c_1 and c_2 (which are the objects of choice) are, as usual:

$$u_1(c_1, c_2) - \lambda = 0$$

$$u_2(c_1, c_2) - \frac{\lambda}{1+r_1} = 0$$

(And of course the FOC with respect to the multiplier just gives back the LBC.) Also as usual, these FOCs can be combined to give the consumption-savings optimality condition,

$\frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} = 1+r_1$. With the given utility function, the marginal utility functions are $u_1 = 1/c_1$ and

$u_2 = 1/c_2$, so the consumption-savings optimality condition in this case becomes $c_2/c_1 = 1+r_1$.

This can be rearranged to give $c_2 = (1+r_1)c_1$, which we can then insert in the LBC to

give $c_1 + c_1 = y_1 + \frac{y_2}{1+r_1}$ (no, that's not a typo, it's $c_1 + c_1$ after the substitution...).

In this problem, you are given neither y_1 nor y_2 . Instead, what you are given is $y_1 + \frac{y_2}{1+r_1} = 26$.

Thus, we have that the optimal quantity of period-1 consumption is $c_1^* = 13$ (which solves part i).

We can **not** compute c_2^* , however, because we are not given the interest rate r_1 (which you would need in order to use the expression $c_2 = (1+r_1)c_1$ computed above. (This solves part ii).

To compute the asset position at the end of period 1, we would need to compute $y_1 - c_1^*$, but since we don't know y_1 , we cannot compute this either (which solves part iii).

Question 1 continued

b. (7 points) To demonstrate how important the concept of the real interest rate is in macroeconomics, an interpretation of it (in addition to the couple of different interpretations we have already discussed in class) is that it reflects the rate of consumption growth between two consecutive periods. Using the consumption-savings optimality condition for the given utility function, describe/discuss whether the real interest rate is **positively related to or negatively related to the rate of consumption growth between two periods**. (Note: No mathematics are especially required for this problem; also note this part can be fully completed even if you were unable to get all the way through part a).

Solution:

The familiar consumption-savings optimality condition is $\frac{u_1}{u_2} = 1 + r$. As we just saw above, for

the given utility function, this becomes $\frac{1/c_1}{1/c_2} = 1 + r$, or, rewriting,

$$\frac{c_2}{c_1} = 1 + r.$$

The left-hand-side of this expression obviously measures the consumption growth rate between period 1 and period 2. That is, if $c_1 = 100$ and $c_2 = 103$, clearly the consumption growth rate is 3 percent between period 1 and period 2. Which would mean that $r = 0.03$. If the real interest rate were instead larger, clearly the left-hand-side, c_2/c_1 , would be larger as well. Thus, **the higher is the real interest rate, the higher is the consumption growth rate between periods – the real interest rate and the consumption growth rate are positively related to each other**.

This is thus yet another way to think about the real interest rate. The two other ways we discussed in class of thinking intuitively about the real interest rate is that it measures the price of current (period-1) consumption in terms of future (period-2) consumption; and as reflecting the fundamental degree of (human) impatience of individuals in the economy. All of these various (and ultimately inter-related) ways of thinking about the real interest underline its fundamental importance in macroeconomic theory.

Note that simply arguing/explaining here that a rise in the real interest rate leads to a fall in period-1 consumption does not address the question – the question is about the **rate of change of consumption between period 1 and period 2**, not about the **level** of consumption in period 1 by itself.

Question 2: Core Inflation and Non-Core Inflation (25 points). Two distinct measures of inflation – called core inflation and non-core inflation – generally garner attention by policy-makers and the media. The core inflation rate is the rate of growth of prices of so-called “core goods” (such as food, clothing, shelter), while the non-core inflation rate is the rate of growth of prices of so-called “non-core goods” (generally energy items).

Consider our usual two-period economy (with no government), in which the representative consumer has no control over his **nominal** income. Rather than there being only one “type” of good the consumer purchases each period, however, suppose that each period there are two “types” of goods: core good and non-core goods. The lifetime utility function of the representative consumer is

$$u(c_1^{CORE}, c_1^{NONCORE}, c_2^{CORE}, c_2^{NONCORE}) = \ln(c_1^{CORE}) + \ln(c_1^{NONCORE}) + \ln(c_2^{CORE}) + \ln(c_2^{NONCORE}),$$

where \ln stands for the natural logarithm. c_1^{CORE} stands for consumption of core goods in period 1, $c_1^{NONCORE}$ stands for consumption of non-core goods in period 1, and similarly for c_2^{CORE} and $c_2^{NONCORE}$.

The representative consumer begins period one with zero assets (i.e., $A_0 = 0$). The period-by-period budget constraints of the representative consumer are thus

$$\begin{aligned} P_1^{CORE} c_1^{CORE} + P_1^{NONCORE} c_1^{NONCORE} + A_1 &= Y_1 \\ P_2^{CORE} c_2^{CORE} + P_2^{NONCORE} c_2^{NONCORE} + A_2 &= Y_2 + (1+i)A_1 \end{aligned}$$

where P_1^{CORE} stands for the **nominal** price of core goods in period 1, $P_1^{NONCORE}$ stands for the **nominal** price of non-core goods in period 1, and similarly for P_2^{CORE} and $P_2^{NONCORE}$. As usual, Y_1 and Y_2 stand for **nominal** income in periods 1 and 2, respectively.

- a. (4 points) Construct, in nominal terms, the representative consumer’s lifetime budget constraint. Show all important steps. Provide **brief** (no more than one sentence) interpretation for the expression you construct.

Solution: Solving the period-2 constraint for A_1 (and using our usual condition $A_2 = 0$), we have

$$A_1 = \frac{P_2^{CORE} c_2^{CORE}}{1+i} + \frac{P_2^{NONCORE} c_2^{NONCORE}}{1+i} - \frac{Y_2}{1+i}.$$

Inserting this in the period-1 constraint and rearranging terms, we have that the LBC is

Question 2a continued (if you need more space)

$$P_1^{CORE} c_1^{CORE} + P_1^{NONCORE} c_1^{NONCORE} + \frac{P_2^{CORE} c_2^{CORE}}{1+i} + \frac{P_2^{NONCORE} c_2^{NONCORE}}{1+i} = Y_1 + \frac{Y_2}{1+i},$$

which has exactly the same interpretation as always: the PDV of all lifetime consumption (which here takes into account both consumption of core and non-core goods) is equal to the PDF of all lifetime income. That is, in a lifetime sense, all consumption spending equals all income regardless of “how many” goods there are to purchase.

A common error here was to define the “inflation rate” in the “usual” way of $P_2/P_1 - 1$. However, note that there are no variables in this model named P_2 or P_1 !....

- b. **(10 points)** Using an appropriate Lagrange formulation (you are free to choose either a lifetime formulation or a sequential formulation), derive **four** optimality conditions: one between period-1 core consumption and period-1 non-core consumption; one between period-2 core consumption and period-2 non-core consumption; one between period-1 core consumption and period-2 core consumption; and one between period-1 non-core consumption and period-2 non-core consumption. Show all important steps.

Solution: Taking the lifetime approach, the Lagrangian is

$$\ln(c_1^{CORE}) + \ln(c_1^{NONCORE}) + \ln(c_2^{CORE}) + \ln(c_2^{NONCORE}) + \lambda \left[Y_1 + \frac{Y_2}{1+i} - P_1^{CORE} c_1^{CORE} - P_1^{NONCORE} c_1^{NONCORE} - \frac{P_2^{CORE} c_2^{CORE}}{1+i} - \frac{P_2^{NONCORE} c_2^{NONCORE}}{1+i} \right]$$

The FOCs are

$$\frac{1}{c_1^{CORE}} - \lambda P_1^{CORE} = 0$$

$$\frac{1}{c_1^{NONCORE}} - \lambda P_1^{NONCORE} = 0$$

$$\frac{1}{c_2^{CORE}} - \frac{\lambda P_2^{CORE}}{1+i} = 0$$

$$\frac{1}{c_2^{NONCORE}} - \frac{\lambda P_2^{NONCORE}}{1+i} = 0$$

Solving out for λ appropriately and then rearranging, we have that the four optimality conditions are, in order,

Question 2b continued (if you need more space)

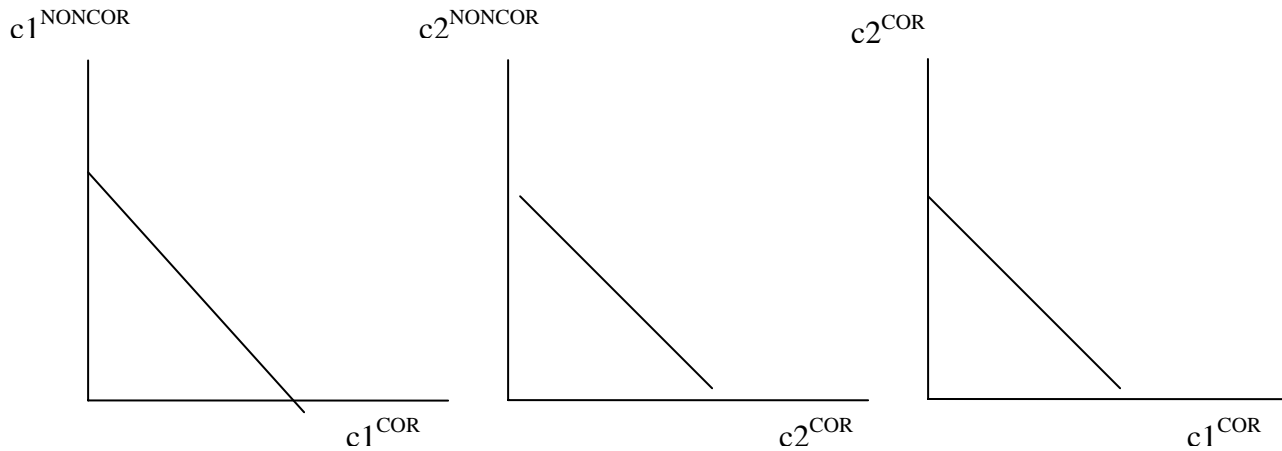
$$\frac{c_1^{NONCORE}}{c_1^{CORE}} = \frac{P_1^{CORE}}{P_1^{NONCORE}}$$

$$\frac{c_2^{NONCORE}}{c_2^{CORE}} = \frac{P_2^{CORE}}{P_2^{NONCORE}}$$

$$\frac{c_2^{CORE}}{c_1^{CORE}} = \frac{P_1^{CORE} (1+i)}{P_2^{CORE}}$$

$$\frac{c_2^{NONCORE}}{c_1^{NONCORE}} = \frac{P_1^{NONCORE} (1+i)}{P_2^{NONCORE}}$$

- c. **(6 points)** In each of the following three diagrams, appropriately label the slope of the budget line. **Briefly** (in no more than one or two sentences) describe how you determined the relevant slopes (you may refer to your work in part b if you need).



Solution: Based on the computed optimality conditions in part b, it is easy to conclude that the slope in the left-most diagram is $-P_1^{CORE} / P_1^{NONCORE}$; the slope in the middle diagram is $-P_2^{CORE} / P_2^{NONCORE}$; and the slope in the right-most diagram is $-P_1^{CORE} (1+i) / P_2^{CORE}$. This follows from the fact that in each of the optimality conditions in part b, the right-hand-side represents the (absolute value of the) slope of the relevant budget line. Alternatively, you could have take the LBC constructed in part a and successively rearranged it to isolate $c_1^{NONCORE}$, $c_2^{NONCORE}$, and c_2^{CORE} , which would give you (in each line, the focus is just on the relevant slop term):

$$c_1^{NONCORE} = -\frac{P_1^{CORE}}{P_1^{NONCORE}} c_1^{CORE} + (\text{other stuff})$$

$$c_2^{NONCORE} = -\frac{P_2^{CORE}}{P_2^{NONCORE}} c_2^{CORE} + (\text{other stuff})$$

$$c_2^{CORE} = -\frac{P_1^{CORE} (1+i)}{P_2^{CORE}} c_1^{CORE} + (\text{other stuff})$$

Question 2 continued

- d. (5 points) Suppose that the Fed changes the nominal interest rate, and suppose that this change in policy does not at all affect P_1^{CORE} , $P_1^{NONCORE}$, P_2^{CORE} , or $P_2^{NONCORE}$, nor does it affect Y_1 or Y_2 . Will this change in monetary policy affect consumers' MRS between period-1 core and period-1 non-core consumption? Will this change in monetary policy affect consumers' MRS between period-2 core and period-2 non-core consumption? For each, **briefly** explain why or why not. You may refer to the diagrams in part c if you need.

Solution: Simply examining the analysis in parts b and c (and with the stated assumption that the monetary policy action has no effect on prices), it is clear that the slopes of the budget lines in the left-most and middle diagrams above will be completely unaffected. Thus, at consumers' optimal choices, their MRS along the period-1 core/period-1 non-core margin and the period-2 core/period-2 non-core margin will be unaffected. The only decisions that will change are the ones **across** periods (that is, the decisions along the period-1 core/period-2 core margin and the period-1 non-core/period-2 non-core margin).

This conclusion highlights the point that monetary policy primarily works by changing the nature of the tradeoff facing individuals in the economy (here, just consumers) **across time**, rather than **within a time period**. That is, monetary policy primarily exerts its effects through intertemporal (across time) channels rather than static (within a time period) channels.

Question 3: International Financial and Macroeconomic Linkages (25 points). Consider a variation of our usual infinite-period “stock-pricing” model. The variation here is that there are two “types” of stock that the representative U.S. consumer can buy: “U.S.” stock and “Foreign” stock. Denote by a_{t-1}^{US} the representative consumer’s holdings of U.S. stock at the beginning of period t and by a_{t-1}^{FOR} the representative consumer’s holdings of foreign stock at the beginning of period t . Likewise, let S_t^{US} and S_t^{FOR} denote, respectively, the nominal price of U.S. and foreign stock in period t , and D_t^{US} and D_t^{FOR} denote, respectively, the per-share nominal dividend that U.S. and foreign stock pay in period t . All of these prices and dividends are in dollar terms. The period- t budget constraint of the representative U.S. consumer is thus

$$P_t c_t + S_t^{FOR} a_t^{FOR} + S_t^{US} a_t^{US} = Y_t + (S_t^{FOR} + D_t^{FOR}) a_{t-1}^{FOR} + (S_t^{US} + D_t^{US}) a_{t-1}^{US},$$

in which all of the other notation is standard: Y_t denotes nominal income (over which the consumer has no control) in period t , c_t is real units of consumption, and P_t is the nominal price of each unit of consumption. To be clear, all of these latter variables refer to U.S.-economy variables. Also as usual, the lifetime utility of the U.S. consumer starting from period t onwards is $u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \dots$, where $\beta \in (0, 1]$ is the usual measure of consumer impatience.

- a. **(8 points)** Setting up an appropriate Lagrangian, derive period- t stock-pricing expressions for both U.S. stock and Foreign stock. Your expressions should be of the form $S_t^{US} = \dots$ and $S_t^{FOR} = \dots$

Solution: Setting up the sequential Lagrangian starting from period t onwards is a straightforward extension of what we’ve seen in class:

$$\begin{aligned} & u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \dots \\ & + \lambda_t \left[Y_t + S_t^{FOR} a_{t-1}^{FOR} + D_t^{FOR} a_{t-1}^{FOR} + S_t^{US} a_{t-1}^{US} + D_t^{US} a_{t-1}^{US} - P_t c_t - S_t^{FOR} a_t^{FOR} - S_t^{US} a_t^{US} \right] \\ & + \beta \lambda_{t+1} \left[Y_{t+1} + S_{t+1}^{FOR} a_t^{FOR} + D_{t+1}^{FOR} a_t^{FOR} + S_{t+1}^{US} a_t^{US} + D_{t+1}^{US} a_t^{US} - P_{t+1} c_{t+1} - S_{t+1}^{FOR} a_{t+1}^{FOR} - S_{t+1}^{US} a_{t+1}^{US} \right] \\ & + \dots \end{aligned}$$

Taking FOCs with respect to c_t , a_t^{FOR} , and a_t^{SP} and combining these FOCs as usual yields two (very similar) stock-pricing equations:

$$\begin{aligned} S_t^{US} &= \frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot (S_{t+1}^{US} + D_{t+1}^{US}) \cdot \frac{P_t}{P_{t+1}} \\ S_t^{FOR} &= \frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot (S_{t+1}^{FOR} + D_{t+1}^{FOR}) \cdot \frac{P_t}{P_{t+1}} \end{aligned}$$

A common error here was to compute the FOC with respect to the variable a_t ...but note that in this model there is no variable named a_t !...

Question 3 continued

Throughout the rest of this question, suppose that **NO** stocks (neither U.S. nor Foreign) **EVER** pay any dividends.

- b. (5 points) Using the two expressions you developed in part a, describe/show how the rate of change (between any two consecutive periods) of U.S. stock prices compares to the rate of change (between the same two consecutive periods) of Foreign stock prices. Clearly show/state all important steps in your argument.

Solution: With $D_t^{US} = D_t^{FOR} = 0$ always (i.e., for any t), the above two expressions can be written as

$$S_t^{US} = \frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot (S_{t+1}^{US}) \cdot \frac{P_t}{P_{t+1}}$$
$$S_t^{FOR} = \frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot (S_{t+1}^{FOR}) \cdot \frac{P_t}{P_{t+1}}$$

Combining these immediately implies that

$$\frac{S_{t+1}^{US}}{S_t^{US}} = \frac{S_{t+1}^{FOR}}{S_t^{FOR}},$$

which shows that the growth rates of US stock prices and Foreign stock prices are equal.

- c. (5 points) Suppose for simplicity that $\beta = 1$ and suppose the economy eventually reaches a steady state. Further suppose that in the steady state, the U.S. inflation rate of consumer goods prices between one period and the next is 5 percent (i.e., $\pi = 0.05$). In the steady state, how quickly do nominal prices of **both** U.S. and Foreign stock grow? Justify your answer with any appropriate combination of mathematical, graphical, or logical arguments.

Solution: In the steady state, all **real** variables become a constant, but not necessarily nominal variables. Thus, imposing steady state on the expressions derived in part a (and using the stated assumptions that $\beta = 1$ and $\pi = 0.05$ and dividends are always zero), we have that

$$\frac{S_{t+1}^{US}}{S_t^{US}} = \frac{S_{t+1}^{FOR}}{S_t^{FOR}} = \frac{P_{t+1}}{P_t} = 1.05.$$

Thus, nominal prices of both U.S. and Foreign stock grow at 5 percent, the same as the consumer-price inflation rate.

The intuition behind this result is as follows. No matter which way we measure the “real interest rate” (whether using U.S. equity returns or Foreign equity returns), they must both be equal to the U.S. consumer’s MRS. Since neither stock pays any dividend, their entire returns must come through changes in the price of the stock itself – i.e., there are capital gains on both U.S. stock and Foreign stock (which is what it means for the price of stock to be growing).

Question 3c continued (if you need more space)

- d. (7 points – Harder) There is also a representative consumer and a central bank in the Foreign country. Continue to focus on just the steady state (of both the U.S. and now also the Foreign economies). If steady-state inflation in the Foreign country is 10 percent (while steady-state inflation in the U.S. remains at 5 percent), how will the growth rates of U.S. stock prices and Foreign stock prices compare with each other? **Here your answer need only be qualitative, rather than a precise numerical value.** Justify your answer with any appropriate combination of mathematical, graphical, or logical arguments. (**Hint:** Your answer should take into account **both** what you found in part c above along with the economic events that must be happening in the Foreign country. Also note there isn't necessarily a single correct analysis here.)

Solution: With a representative consumer in the Foreign country, you could do exactly the same sort of analysis as in parts a through c for the Foreign representative consumer; you would arrive at the conclusion

$$\frac{S_{t+1}^{US}}{S_t^{US}} = \frac{S_{t+1}^{FOR}}{S_t^{FOR}} = \frac{P_{t+1}^{FOR}}{P_t^{FOR}} = 1.1,$$

where P_t^{FOR} and P_{t+1}^{FOR} denote the period-t and period-t+1 price levels in the Foreign country.

From this, you would want to conclude that both U.S. and Foreign stock prices grow at 10 percent. But, since you were told to take account both implications of events abroad and the implications (derived in part c) of events in the U.S., there is an inconsistency...namely, the analysis in part c leads one to conclude that stock prices everywhere are growing at 5 percent per year, whereas the analysis here leads one to conclude that stock prices everywhere are growing at 10 percent per year.

In some sense, this was perhaps the most important conclusion to be able to reach here. The rest of your analysis was meant to make you think about issues rather than just models. Your qualitative analysis as to how to “reconcile” this apparent contradiction could have taken into account any of a number of many things: due to some sorts of cross-border restrictions on capital (stock) holdings, perhaps the prices of stocks would not be equated across countries in any case; even absent such capital restrictions, the returns on assets would *somehow* be equalized due to trade (of actual goods and services) considerations, etc. The model(s) we have developed thus far have not at all made any international linkages explicit, so your analysis here was necessarily ideas-based; but, in the end, ideas are what formal economic models try to articulate.

Question 4: Welfare Losses from Distortionary Taxation (30 points). A central question behind many policy proposals in macroeconomics and public finance is the potential welfare consequences of distortionary tax systems. Here you will consider the welfare consequences associated with labor income taxation in the context of the **static consumption-leisure model**. Specifically, we will consider how much consumers “lose” due to a labor income tax rate of 20 percent as compared to a labor income tax rate of 0 (zero) percent. (To connect with ideas you know from basic economics, what you will formally compute is the “deadweight loss” of a system with 20-percent labor income taxation compared to a system with zero-percent labor income taxation.)

A typical welfare analysis proceeds in the following three steps:

1. Solve for the representative consumer’s optimal choices under the proposed tax policy (in our case, a labor income tax rate of 20 percent).
2. Solve for the representative consumer’s optimal choices under the alternative tax policy (in our case, a labor income tax rate of zero percent).
3. **Compute the ADDITIONAL PERCENTAGE OF CONSUMPTION the representative consumer would need under the proposed tax policy to be EQUALLY HAPPY IN TERMS OF UTILITY as under the alternative tax policy.**

How to implement step 3 is described in detail after part b below.

Recall that, in the static consumption-leisure model, the representative consumer’s budget constraint can be expressed in real terms as $c + (1-t)wl = 168(1-t)w$; for simplicity, suppose throughout the analysis that the real wage is $w = 1$ no matter what tax policy is in place.

A critical component to the answer to welfare questions can be what the precise functional form of the (representative consumer’s) utility function is.

- a. **(7 points)** Suppose the representative consumer’s utility function over consumption and leisure is $u(c, l) = \ln c + \ln l$. Numerically solve for the optimal choices of consumption and leisure under **both** the proposed policy (20 percent labor tax) and the alternative policy (zero labor tax) – that is, conduct step 1 and step 2 of the analysis for this utility function.

Solution: We know that optimal choice is described by the consumption-leisure optimality condition, $\frac{u_l}{u_c} = (1-t)w$. With $w = 1$ and the given utility function, we have

$\frac{u_l}{u_c} = \frac{1/l}{1/c} = \frac{c}{l} = (1-t)$, which obviously implies that at the optimal choice (asterisks denoting optimal choice are omitted here), $c = (1-t)l$. Inserting this expression for c back into the budget constraint, we have that

$$(1-t)l + (1-t)l = 168(1-t),$$

Question 4a continued (if you need more space)

which can be simplified to $2l = 168$. It immediately follows that the optimal choice of leisure is $l^* = 84$, which of course means $n^* = 168 - l^* = 84$. **Note that this optimal choice of leisure is completely independent of the labor income tax rate.**

Next, to complete step 1 of the analysis, we must compute c^* under the proposed policy. We found above that at the optimal choice, $c = (1-t)l$; hence, $c^* = (1-t)l^* = (1-0.2)84 = 67.2$. Step 1 of the analysis is complete.

Proceeding to step 2: we noted above that the optimal choice of leisure (and labor) was independent of the tax rate (this is because the analysis that led us to $l^* = 84$ was true for **any** labor income tax rate). Hence, under the alternative policy we still have $l^* = 84$. Now we must compute c^* under the alternative policy: again, use the optimality condition to determine $c^* = (1-t)l^* = (1-0)84 = 84$. Step 2 of the analysis is complete.

- b. **(8 points)** Suppose the representative consumer's utility function over consumption and leisure is $u(c, l) = 2\sqrt{c} + 2\sqrt{l}$. Numerically solve for the optimal choices of consumption and leisure under **both** the proposed policy (20 percent labor tax) and the alternative policy (zero labor tax) – that is, conduct step 1 and step 2 of the analysis for this utility function.

Solution: Once again, use $w = 1$ and the fact that optimal choice is described by the consumption-leisure optimality condition, $\frac{u_l}{u_c} = (1-t)w$. For the given utility function, we have

$$\frac{u_l}{u_c} = \frac{1/\sqrt{l}}{1/\sqrt{c}} = \frac{\sqrt{c}}{\sqrt{l}} = (1-t),$$
 which obviously implies that at the optimal choice (asterisks denoting

optimal choice are omitted here), $c = (1-t)^2 l$. (Note the square term here.) Inserting this expression for c back into the budget constraint, we have that

$$(1-t)^2 l + (1-t)l = 168(1-t),$$

which can be simplified to $l[1+1-t] = 168$. It immediately follows that the optimal choice of leisure is $l^* = 168/(2-t)$, which now **does depend on the labor income tax rate**, unlike in part a.

Thus, under the proposed policy, $l^* = 168/(2-0.2) = 93.333$. To complete step 1 of the analysis, we must compute c^* under the proposed policy. We found above that at the optimal choice, $c = (1-t)^2 l$; hence, $c^* = (1-t)^2 l^* = (1-0.2)^2 93.333 = 59.7333$. Step 1 of the analysis is complete.

Proceeding to step 2: with zero taxation, $l^* = 168/(2-0) = 84$. Now we must compute c^* under the alternative policy: again, use the optimality condition to determine $c^* = (1-t)^2 l^* = (1-0)84 = 84$. Step 2 of the analysis is complete.

Question 4 continued

(NOTE: The remainder of this question is more difficult.)

Implementing step 3 of the welfare analysis means first comparing the total number of **utils** under the proposed policy with the total number of **utils** under the alternative tax policy and then determining how much **additional consumption (measured in percent)** would be required under the proposed policy to make the total number of utils the **same** under both policies.

For example, suppose the total number of utils (when consumers are making their optimal consumption-leisure choices) under the proposed policy were 5, while the total number of utils (when consumers are making their optimal consumption-leisure choices) under the alternative policy were 7. If policy-makers implemented the proposed policy and also could somehow just “hand” to consumers an extra X percent of consumption that would make them equally well off (in utility terms) as under the alternative policy, the number X would measure the welfare loss of the proposed policy compared to the alternative policy. The reason why welfare losses are computed this way is because, as you recall from basic economics, utils in and of themselves are meaningless numbers.

Mathematically, the welfare measure X solves the equation

$$u\left((1+X)c_{PROPOSED\ POLICY}^*, l_{PROPOSED\ POLICY}^*\right) = u\left(c_{ALTERNATIVE\ POLICY}^*, l_{ALTERNATIVE\ POLICY}^*\right);$$

note carefully how the variable X enters this expression.

- c. (7 points) Based on your work in part a, compute, for the utility function $u(c, l) = \ln c + \ln l$, how much extra consumption X (as defined above) is required under the tax policy of 20 percent as compared to the tax policy of zero percent? Show all important steps/logic in your work. (Note: the number X will not necessarily be a nice “round” number and may require calculator-work.)

Solution: For step 3 using the log-log utility function, compute the total number of utils under the proposed and alternative policies:
 $u\left(c_{PROPOSED\ POLICY}^*, n_{PROPOSED\ POLICY}^*\right) = u(67.2, 84) = \ln(67.2) + \ln(84) = 8.6385$, and
 $u\left(c_{ALTERNATIVE\ POLICY}^*, n_{ALTERNATIVE\ POLICY}^*\right) = u(84, 84) = \ln(84) + \ln(84) = 8.8616$. Finally, we must compute the welfare measure X that makes utility the same under the two policies.

Clearly, the only way to make utility the same under the two policies is to increase the first argument (ie, the consumption argument) of the utility function to 84 under the proposed policy (since the leisure argument is 84 under both policies anyway): ie, increasing 67.2 to 84 would make consumers just as happy under either policy. Thus, we need $(1+X)67.2 = 84$, which means $X = 0.25$; the interpretation is that if consumers were “given” 25 percent more consumption in the world with a 20-percent income tax, they would be just as well off (in utility terms) as in the world with no tax.

Question 4c continued (if you need more space)

- d. **(8 points)** Based on your work in part a, compute, for the utility function $u(c, l) = 2\sqrt{c} + 2\sqrt{l}$, how much extra consumption X (as defined above) is required under the tax policy of 20 percent as compared to the tax policy of zero percent? Show all important steps/logic in your work. (**Note:** the number X will not necessarily be a nice “round” number and may require calculator-work.)

Solution: For step 3 using the square-root utility function, compute the total number of utils under the proposed and alternative policies:
 $u(c_{PROPOSED\ POLICY}^*, n_{PROPOSED\ POLICY}^*) = u(59.733, 93.333) = 2\sqrt{59.733} + 2\sqrt{93.333} = 34.7792$, and
 $u(c_{ALTERNATIVE\ POLICY}^*, n_{ALTERNATIVE\ POLICY}^*) = u(84, 84) = 2\sqrt{84} + 2\sqrt{84} = 36.6606$. Finally, we must compute the welfare measure X that makes utility the same under the two policies. The mathematical problem here is a bit harder than in part c above because we cannot simply conclude that “all we must do is raise the first argument to 84” to make consumers equally well off under the two policies – because the **second** arguments (the leisure arguments) are not identical as they were in the log-log utility function.

Question 4d continued (if you need more space)

Thus, the computational problem boils down to figuring out the X that makes the following equation true:

$$2\sqrt{(1+X)59.733} + 2\sqrt{93.333} = 2\sqrt{84} + 2\sqrt{84} = 36.6606.$$

Simplifying these computations,

$$2\sqrt{(1+X)59.733} = 36.6606 - 2\sqrt{93.333},$$

which means

$$\sqrt{(1+X)59.733} = \frac{36.6606 - 2\sqrt{93.333}}{2} = 8.6694.$$

Squaring both sides, $(1+X)59.733 = 8.6694^2 = 75.1585$, from which we can obtain $X = 0.2582$. The interpretation is that if consumers were “given” about 26 percent more consumption in the world with a 20-percent income tax, they would be just as well off (in utility terms) as in the world with no tax.

Note that this answer differs from what we obtained in part c, which shows that what policy-makers believe/assume about the functional form of consumer utility functions can matter for determining **how** (quantitatively) “good” or “bad” a particular policy is.