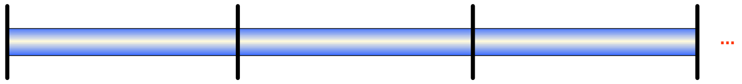
INFINITE-PERIOD ECONOMY (CONTINUED)

MARCH 2, 2009

BASICS

- Timeline of events



- Notation
  - c_t: consumption in period *t*
  - P_t: nominal price of consumption in period *t*
  - Y_t: nominal income in period *t* (“falls from the sky”)
  - a_{t-1}: real wealth (stock) holdings at beginning of period *t* / end of period *t*-1
  - S_t: nominal price of a unit of stock in period *t*
  - D_t: nominal dividend paid in period *t* by each unit of stock held at the start of *t*

The “definining features” of stock

  - π_{t+1}: net inflation rate between period *t* and period *t*+1

$$\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t} \left(= \frac{P_{t+1}}{P_t} - 1 \right)$$

  - y_t: real income in period *t* (= Y_t / P_t)

BASICS

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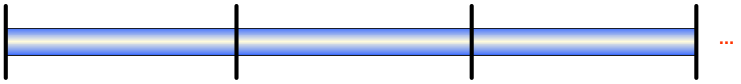

- ☐ **Notation**
  - ☐ c_{t+1}: consumption in period t+1
  - ☐ P_{t+1}: nominal price of consumption in period t+1
  - ☐ Y_{t+1}: nominal income in period t+1 (“falls from the sky”)
  - ☐ a_t: real wealth (stock) holdings at beginning of period t+1 / end of period t
  - ☐ S_{t+1}: nominal price of a unit of stock in period t+1
  - ☐ D_{t+1}: nominal dividend paid in period t by each unit of stock held at the <u>start</u> of t+1
  - ☐ π_{t+2}: net inflation rate between period t+1 and period t+2

$$\pi_{t+2} = \frac{P_{t+2} - P_{t+1}}{P_{t+1}} \left(= \frac{P_{t+2}}{P_{t+1}} - 1 \right)$$

  - ☐ y_{t+1}: real income in period t+1 (= Y_{t+1} / P_{t+1})

The “definining features” of stock

a_{t-1}

Economic events during period t: income, consumption, savings

a_t

Economic period t consump

Period t

Peri

BASICS

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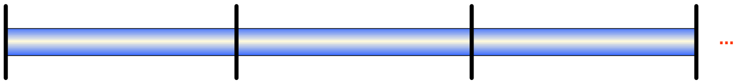

- ☐ **Notation**
  - ☐ And so on for period t+2, t+3, etc…

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

- **Step 1: Construct Lagrange function (starting from *t*)**

IMPORTANT: Discount factor β multiplies ***both*** future utility ***and*** future budget constraints

Everything (utility ***and*** income) about the future is discounted

$$u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})+\beta^3 u(c_{t+3})+\dots$$
First the lifetime utility function….

$$+\lambda_t\left[Y_t+(S_t+D_t)a_{t-1}-P_tc_t-S_ta_t\right]$$
…then the period *t* constraint…

$$+\beta\lambda_{t+1}\left[Y_{t+1}+(S_{t+1}+D_{t+1})a_t-P_{t+1}c_{t+1}-S_{t+1}a_{t+1}\right]$$
…then the period *t*+1 constraint…

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…then the period *t*+2 constraint…

$$+\beta^3\lambda_{t+3}\left[Y_{t+3}+(S_{t+3}+D_{t+3})a_{t+2}-P_{t+3}c_{t+3}-S_{t+3}a_{t+3}\right]$$
…then the period *t*+3 constraint…

$$+\dots$$
Infinite number of terms

- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

with respect to c_t:

with respect to a_t:

with respect to c_{t+1}:

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- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

Identical except for time subscripts

with respect to c_t: $u'(c_t)-\lambda_tP_t=0$ Equation 1

with respect to a_t: $-\lambda_tS_t+\beta\lambda_{t+1}(S_{t+1}+D_{t+1})=0$ Equation 2 – the basis for asset pricing theories

with respect to c_{t+1}: $\cancel{\beta}u'(c_{t+1})-\cancel{\beta}\lambda_{t+1}P_{t+1}=0$ Equation 3

Finance Fundamentals

THE BASICS OF ASSET PRICING

$$u'(c_t) - \lambda_t P_t = 0$$ Equation 1

$$-\lambda_t S_t + \beta \lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ Equation 2

$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0$$ Equation 3

- **Equation 2 →** $S_t = \left(\frac{\beta \lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$ BASIC ASSET-PRICING EQUATION

Period-*t* stock price = Pricing kernel x Future return

Two components:
1. Future price of stock
2. Future dividend payment

- **Much of finance theory concerned with pricing kernel**
  - **Theoretical properties**
  - **Empirical models of kernels**

- **Pricing kernel where macro theory and finance theory intersect**
  - **Solve equations 1 and 3 for λ_t and λ_{t+1}**
  - **Insert in asset-pricing equation**

Macro-Finance Connections

MACROECONOMIC EVENTS AFFECT ASSET PRICES

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

- **Consumption across time (c_t and c_{t+1}) affects stock prices**
  - **Fluctuations over time in aggregate consumption impact S_t**

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- ***Wall Street Journal* page A1, Feb. 21, 2008:**

  "Stocks [prices] fell on the [higher-than-expected] inflation reading."

  Interpretation: Increase in π_t (which typically would also be associated with increase in π_{t+1}) led to fall in S_t.

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- ☐ **Inflation affects stock prices**
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- ☐ ***Wall Street Journal* page A1, Oct. 3, 2008:**

  "Stock prices fell on fears of short-run weakness in the economy."

  Interpretation: A decrease in c_t (induced by the "credit crunch") makes stocks *right now* (i.e., in period *t*) a less attractive asset – demand for it falls, hence its price (S_t) falls.

MACROECONOMIC EVENTS AFFECT ASSET PRICES

Combining equations 1, 2, and 3 $\rightarrow$

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VIEW AS A CONSUMPTION-SAVINGS OPTIMALITY CONDITION $\rightarrow$

- ☐ **Consumption across time (c_t and c_{t+1}) affects stock prices**
  - ☐ **Fluctuations over time in aggregate consumption impact S_t**
- ☐ **Inflation affects stock prices**
  - ☐ **Fluctuations over time in inflation impact S_t**
- ☐ **ANY factor (monetary policy, fiscal policy, globalization, etc.) that affects inflation and GDP in principle impacts stock/asset markets**
- ☐ **Direction of causality?...**

CONSUMER OPTIMIZATION

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1}+D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

↓ **Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side**

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$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS between period } t \text{ consumption and period } t+1 \text{ consumption}} = \underbrace{\left(\frac{S_{t+1}+D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)}_{\text{Some sort of price ratio...}}$$

i.e., ratio of marginal utilities →

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CONSUMPTION-SAVINGS OPTIMALITY CONDITION

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MRS between period *t* consumption and period *t*+1 consumption

Analogy with Chapters 3 & 4: must be (1+r_t)

← **Recall real interest rate is a *price***

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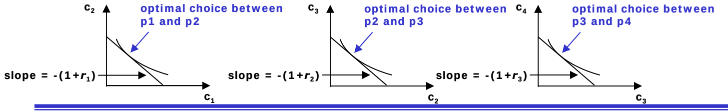
- **Recover Chapter 3 & 4 model by setting t =1 and β = 1**
- **Infinite-period model is sequence of *overlapping* two-period models**



etc.

A LONG-RUN THEORY OF MACRO

- **Consumption-savings optimality condition at the heart of modern macro models**
  - **Emphasizes the dynamic nature of aggregate economic events**
  - **Foundation for understanding the periodic ups and downs (“business cycles”) of the economy**
  - **(Chapter 13: business cycle theory)**

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1+r_t$$

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NEXT: Impose “steady state” and examine long-run relationship between interest rates and consumer impatience

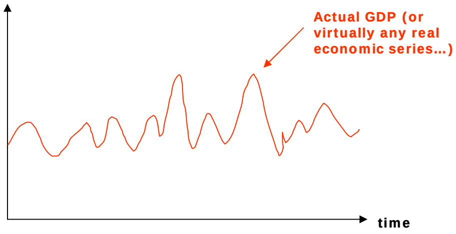
$$\frac{1}{\beta} = 1 + r$$

STEADY-STATE (LONG-RUN) OF INFINITE-PERIOD ECONOMY: WHY ARE INTEREST RATES POSITIVE?

MARCH 2, 2009

A LONG-RUN THEORY OF MACRO

- Aggregate economic activity tends to "settle down eventually"



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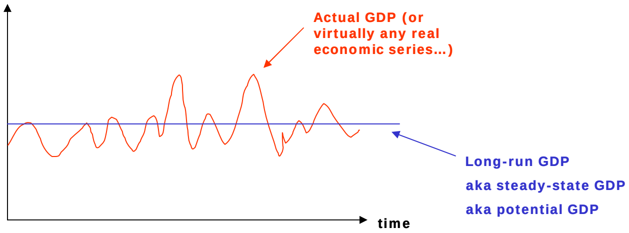

  - The "ups and downs" are business cycles
  - The "average" is the long-run
    - Technical terminology: steady-state
  - Business-cycle theory after midterm exam

STEADY STATE

- **Steady state**
  - **A concept from differential equations**
  - **(Optimality conditions of economic models are differential equations…)**
  - **Heuristic definition: in a dynamic (mathematical) system, a steady-state is a condition in which the variables that are moving over time settle down to constant values**

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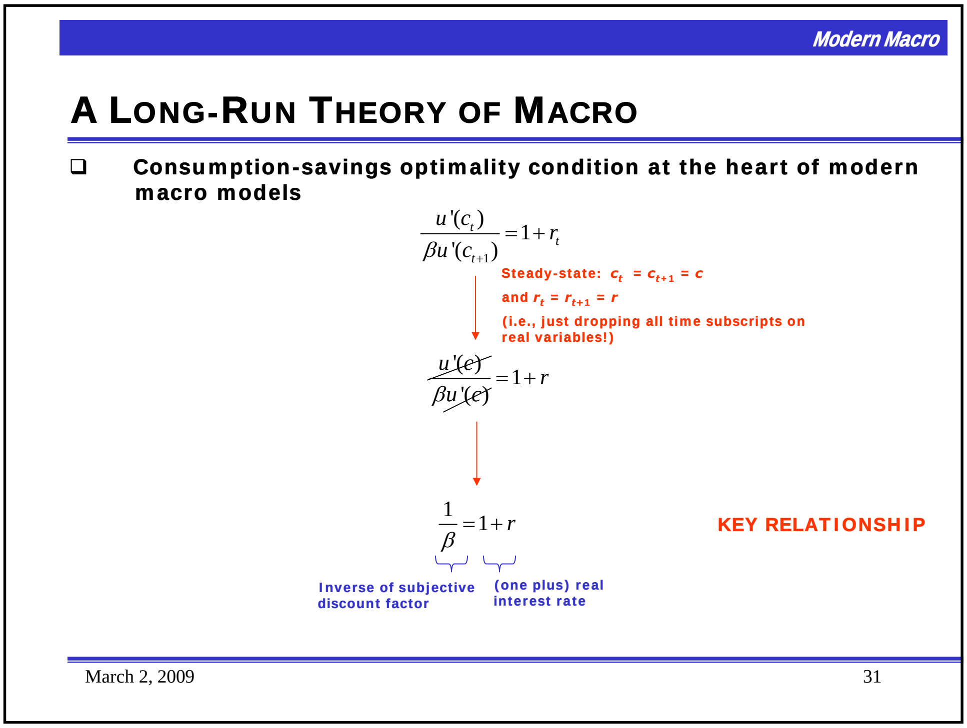

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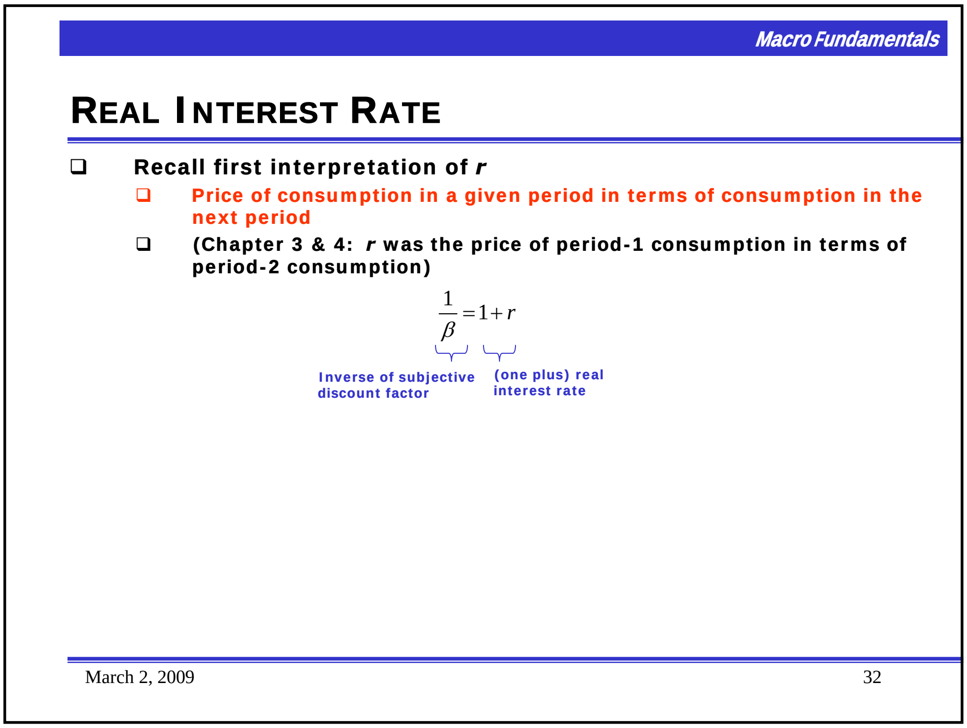
$$\underbrace{\frac{1}{\beta}}_{\text{Inverse of subjective discount factor}} = \underbrace{1 + r}_{\text{(one plus) real interest rate}}$$

KEY RELATIONSHIP



REAL INTEREST RATE

- ❑ **Recall first interpretation of *r***
  - ❑ **Price of consumption in a given period in terms of consumption in the next period**
  - ❑ **(Chapter 3 & 4: *r* was the price of period-1 consumption in terms of period-2 consumption)**

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- Which came first, β or *r*?
  - Modern macro view: $\beta < 1$ causes $r > 0$, not the other way around
  - A deep view of ***why*** positive interest rates exist in the world