LINEARIZING AND APPROXIMATING THE RBC MODEL

SEPTEMBER 9, 2008

LINEARIZATION

- **For** $f(x,y,z)=0$**, multivariable Taylor linear expansion around** $(\bar{x},\bar{y},\bar{z})$

$$f(x,y,z)\approx f(\bar{x},\bar{y},\bar{z})+f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x})+f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y})+f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$

LINEARIZATION OF THE RBC MODEL

- **For $f(x,y,z)=0$, multivariable Taylor linear expansion around $(\bar{x},\bar{y},\bar{z})$**

$$f(x,y,z) \approx f(\bar{x},\bar{y},\bar{z}) + f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x}) + f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y}) + f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$

- **Four equations describe the dynamic solution to RBC model**
  - **Consumption-leisure efficiency condition**

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)} = z_t m_n(k_t,n_t)$$

  - **Consumption-investment efficiency condition**

$$u_c(c_t,n_t) = \beta E_t\left[u_c(c_{t+1},n_{t+1})\left(1-\delta+z_{t+1}m_k(k_{t+1},n_{t+1})\right)\right]$$

  - **Aggregate resource constraint**

$$c_t + k_{t+1} - (1-\delta)k_t = z_t m(k_t,n_t)$$

  - **Law of motion for TFP**

$$\ln z_{t+1} = (1-\rho_z)\ln\bar{z} + \rho_z \ln z_t + \varepsilon^z_{t+1}$$

STEADY STATE

- **Deterministic steady state the natural local point of approximation**
- **Shut down all shocks and set exogenous variables at their means**
- **The Idea: Let economy run for many (infinite) periods**
  - **Time eventually "doesn't matter" any more**
  - **Drop all time indices**

$$-\frac{u_n(\bar{c},\bar{n})}{u_c(\bar{c},\bar{n})} = \bar{z}m_n(\bar{k},\bar{n})$$

$$u_c(\bar{c},\bar{n}) = \beta u_c(\bar{c},\bar{n})\left[m_k(\bar{k},\bar{n})+1-\delta\right]$$

$$\bar{c} + \delta\bar{k} = \bar{z}f(\bar{k},\bar{n})$$

$$\ln\bar{z} = (1-\rho_z)\ln\bar{z} + \rho_z\ln\bar{z} \Rightarrow \bar{z} = \bar{z} \quad \text{(a parameter of the model)}$$

- **Given functions and parameter values (next…), solve for (*c*, *n*, *k*)**
  - **The steady-state of the model**
  - **Taylor expansion around this point**

LINEARIZATION ALGORITHMS

- **Schmitt-Grohe and Uribe (2004 *JEDC*)**
  - **A perturbation algorithm**
    - **A class of methods used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem.**
    - **Applicable if the problem can be formulated by adding a “small” term to the description of the exactly-solvable problem.**
  - **Matlab code available through Duke economics web site**

- **Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)**
  - **Uses a generalized eigen-decomposition**
    - **Typically implemented with Schur decomposition (Sims algorithm)**
  - **Matlab code available at**

    http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm

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LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model's dynamic equations in a vector $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition	
Consumption-investment efficiency condition	
Aggregate resource constraint	
Law of motion for TFP	

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

1. **Differentiate** $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_{t+1}

First derivatives with respect to:

	c_{t+1}	n_{t+1}
Consumption-leisure efficiency condition		
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

$= f_{y_{t+1}}$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

2. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_t

First derivatives with respect to:

$$\begin{array}{l|cc|}
 & c_t & n_t \\
\text{Consumption-leisure efficiency condition} & & \\
\text{Consumption-investment efficiency condition} & & \\
\text{Aggregate resource constraint} & & \\
\text{Law of motion for TFP} & &
\end{array} = f_{y_t}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

3. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_{t+1}

First derivatives with respect to:

$$\begin{array}{l|cc|}
 & k_{t+1} & z_{t+1} \\
\text{Consumption-leisure efficiency condition} & & \\
\text{Consumption-investment efficiency condition} & & \\
\text{Aggregate resource constraint} & & \\
\text{Law of motion for TFP} & &
\end{array} = f_{x_{t+1}}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

4. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_t

First derivatives with respect to:

$$\begin{array}{l|cc|} & k_t & z_t \\ \text{Consumption-leisure efficiency condition} & & \\ \text{Consumption-investment efficiency condition} & & \\ \text{Aggregate resource constraint} & & \\ \text{Law of motion for TFP} & & \end{array} = f_{x_t}$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = E_t \begin{bmatrix} f^1(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^2(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^3(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^4(y_{t+1}, y_t, x_{t+1}, x_t) \end{bmatrix} \begin{array}{l} \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{array}$$

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Consumption-leisure efficiency condition
Consumption-investment efficiency condition
Aggregate resource constraint
Law of motion for TFP

Conjecture equilibrium decision rules

$$y_t=g(x_t,\sigma)$$
$$x_{t+1}=h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

"Perturbation parameter": governs size of shocks

Matrix of standard deviations of state variables

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Consumption-investment efficiency condition
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Conjecture equilibrium decision rules

$$y_t=g(x_t,\sigma)$$
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"Perturbation parameter": governs size of shocks

Matrix of standard deviations of state variables

Substitute decision rules into dynamic equations

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$
\begin{aligned}
&E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0 \\
&\qquad = E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right] \\
&\qquad = E_t\left[f(g(h(x_t,\sigma),\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right] \\
&\qquad \equiv F(x_t,\sigma)
\end{aligned}
$$

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\end{aligned}
$$

$$\Downarrow$$

$$F_x(x_t,\sigma) = 0 \qquad\qquad F_\sigma(x_t,\sigma) = 0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$
\begin{aligned}
E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]&=0\\
&=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]\\
&=E_t\left[f(g(h(x_t,\sigma),\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]\\
&\equiv F(x_t,\sigma)
\end{aligned}
$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$
\begin{aligned}
F_x(x_t,\sigma)&=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}\\
&=0
\end{aligned}
$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$
\begin{aligned}
E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]&=0\\
&=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]\\
&=E_t\left[f(g(h(x_t,\sigma),\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]\\
&\equiv F(x_t,\sigma)
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Using chain rule and suppressing arguments

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F_x(x_t,\sigma)&=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}\\
&=0
\end{aligned}
$$

Setting $\sigma = 0$ shuts down shocks

Holds, in particular, at the deterministic steady state $(\bar{x},0)$

$$F_x(\bar{x},0)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_x=0$$

Each term is evaluated at the steady state – just as Taylor theorem requires

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x**

$$F_x(\bar{x},0) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t} = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

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- **First-order approximation is**

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + g_\sigma(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + h_\sigma(\bar{x},0)\sigma$$

LINEARIZATION OF THE RBC MODEL

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- **Solve numerically for the elements of g_x and h_x (use `fsolve` in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t, \sigma)$$
$$x_{t+1} = h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}$$

- **First-order approximation is**

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + \overset{= 0}{g_\sigma}(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + \underset{= 0}{h_\sigma}(\bar{x},0)\sigma$$

SGU Theorem 1:
$g_\sigma = 0$ and $h_\sigma = 0$

LINEARIZATION OF THE RBC MODEL

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$$F_x(\bar{x},0) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t} = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` in Matlab)**
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SGU Theorem 1:
$g_\sigma = 0$ and $h_\sigma = 0$

- **DONE!!!**
  - **Now conduct impulse responses, run simulations, tabulate moments, write paper**

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation (i.e., the "size") of the exogenous shocks**
- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem (Ljungqvist and Sargent (2004, p. 113))**

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- **Here, we have**

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + \overset{= 0}{\cancel{g_\sigma}}(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + \underset{= 0}{\cancel{h_\sigma}}(\bar{x},0)\sigma$$

- **SGU Theorem 1: $g_\sigma = 0$ and $h_\sigma = 0$**
  - **First-order approximated decision rules do not depend on the size of the shocks, which is governed by σ**
  - **Not quite the same thing as CE, but we'll loosely refer to it as CE**

SGU Details

SECOND-ORDER APPROXIMATION

- **Use computed g_x and h_x to construct linear system of equations needed to solve for coefficients of *second-order* approximation (SGU p. 762-763)**

$$
\begin{aligned}
[F_{xx}(\bar{x},0)]^i_{jk} = &([f_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k \\
&+ [f_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{y'x}]^i_{\alpha k})[g_x]^\alpha_\beta[h_x]^\beta_j \\
&+ [f_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j \\
&+ [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk} \\
&+ ([f_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{yx}]^i_{\alpha k})[g_x]^\alpha_j \\
&+ [f_y]^i_\alpha[g_{xx}]^\alpha_{jk} \\
&+ ([f_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [f_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k + [f_{x'x}]^i_{\beta k})[h_x]^\beta_j \\
&+ [f_{x'}]^i_\beta[h_{xx}]^\beta_{jk} \\
&+ [f_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [f_{xx'}]^i_{j\delta}[h_x]^\delta_k + [f_{xx}]^i_{jk} \\
= 0;& \quad i = 1,\ldots,n, \quad j,k,\beta,\delta = 1,\ldots,n_x; \quad \alpha,\gamma = 1,\ldots,n_y.
\end{aligned}
$$

$$
\begin{aligned}
[F_{\sigma\sigma}(\bar{x},0)]^i = &[f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{\sigma\sigma}]^\beta \\
&+ [f_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[\eta]^\delta_\xi[g_x]^\alpha_\beta[\eta]^\beta_\phi[I]^\phi_\xi \\
&+ [f_{y'x'}]^i_{\alpha\delta}[\eta]^\delta_\xi[g_x]^\alpha_\beta[\eta]^\beta_\phi[I]^\phi_\xi \\
&+ [f_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[\eta]^\delta_\xi[\eta]^\beta_\phi[I]^\phi_\xi \\
&+ [f_{y'}]^i_\alpha[g_{\sigma\sigma}]^\alpha \\
&+ [f_y]^i_\alpha[g_{\sigma\sigma}]^\alpha \\
&+ [f_{x'}]^i_\beta[h_{\sigma\sigma}]^\beta \\
&+ [f_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[\eta]^\delta_\xi[\eta]^\beta_\phi[I]^\phi_\xi \\
&+ [f_{x'x'}]^i_{\beta\delta}[\eta]^\delta_\xi[\eta]^\beta_\phi[I]^\phi_\xi \\
= 0;& \quad i = 1,\ldots,n; \quad \alpha,\gamma = 1,\ldots,n_y; \quad \beta,\delta = 1,\ldots,n_x; \\
&\phi,\xi = 1,\ldots,n_\epsilon.
\end{aligned}
$$

- **SGU Theorem 1: $g_{x\sigma} = 0$ and $h_{x\sigma} = 0$; but "certainty equivalence" does not hold because** $g_{\sigma\sigma} \neq 0, h_{\sigma\sigma} \neq 0$

(Partial) Example

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} = (1-\alpha) z_t k_t^\alpha n_t^{-\alpha}$

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- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} = (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha}$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**

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- **Compute first row of matrix** $f_{y_{t+1}}$

$$\begin{array}{l|cc|} & c_{t+1} & n_{t+1} \\ \hline \textbf{Consumption-leisure efficiency condition} & 0 & 0 \\ \textbf{Consumption-investment efficiency condition} & & \\ \textbf{Aggregate resource constraint} & & \\ \textbf{Law of motion for TFP} & & \end{array}$$

(Partial) Example

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- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$**)**
- **Compute first row of matrix** f_{yt}

$$\begin{array}{l c} & \begin{array}{cc} \boldsymbol{c_t} & \boldsymbol{n_t} \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \begin{bmatrix} \dfrac{\psi}{n_t} & -\dfrac{\psi c_t}{n_t^2} + \alpha(1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha-1} \\ & \\ & \\ & \end{bmatrix} \end{array}$$

(Partial) Example

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- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} = (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha}$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$**)**
- **Compute first row of matrix** f_{xt+1}

$$\begin{array}{l c} & \begin{array}{cc} \boldsymbol{k_{t+1}} & \boldsymbol{z_{t+1}} \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \begin{bmatrix} 0 & 0 \\ & \\ & \\ & \end{bmatrix} \end{array}$$

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} = (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha}$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix** f_{xt}

$$
\begin{array}{l|cc}
 & \boldsymbol{k_t} & \boldsymbol{z_t} \\
\textbf{Consumption-leisure efficiency condition} & -\alpha(1-\alpha) z_t \dfrac{k_t^{\alpha-1}}{n_t^{\alpha}} & -(1-\alpha) \dfrac{k_t^{\alpha}}{n_t^{\alpha+1}} \\
\textbf{Consumption-investment efficiency condition} & & \\
\textbf{Aggregate resource constraint} & & \\
\textbf{Law of motion for TFP} & &
\end{array}
$$

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of** $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ **are**

f_{yt+1}	0	0
f_{yt}	$\dfrac{\psi}{\bar{n}}$	$-\dfrac{\psi \bar{c}}{\bar{n}^2} + \alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\bar{z}\dfrac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}}$	$-(1-\alpha)\dfrac{\bar{k}^{\alpha}}{\bar{n}^{\alpha+1}}$

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ are**

f_{yt+1}	0	0
f_{yt}	$\dfrac{\psi}{\overline{n}}$	$-\dfrac{\psi\overline{c}}{\overline{n}^2}+\alpha(1-\alpha)\overline{z}\overline{k}^{\alpha}\overline{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\overline{z}\dfrac{\overline{k}^{\alpha-1}}{\overline{n}^{\alpha}}$	$-(1-\alpha)\dfrac{\overline{k}^{\alpha}}{\overline{n}^{\alpha+1}}$

- **How to compute derivatives?**
  - **By hand (feasible for small models)**
  - **Schmitt-Grohe and Uribe Matlab analytical routines**
  - **Your own Maple or Mathematica programs**
  - **Dynare package**